Summary of

Workbook Unit 6:

Tautologies, Contradictions and Contingencies

This is only a summary of Unit 6, which is intended as a general overview of the unit. This text does **not** include full explanations or examples or exercises.

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1. Four Properties of Propositions

Propositions can be contingently or logically true or contingently or logically false.

1.1. Contingent and Logical Truth

Some propositions are contingently true (e.g. The President of the U.S.A. is a man), some are logically true (It will either rain or not rain in Hattiesburg on June 28, 20). The truth of contingently true statements depends on how things are in the world. The President of the U.S.A. is, *as a matter of fact*, a man but facts could be otherwise.

Logically true propositions are true necessarily, independently of the facts. The claim "It will either rain or not rain in Hattiesburg on June 28, 2020" is true independently of how the facts turn out. Logically true propositions are true *in virtue of their logical form*. This means that any proposition that shares the same logical form with a given logically true proposition will also be true, in fact will be logically true. The logical form of the proposition "It will either rain or not rain in Hattiesburg on June 28, 2020" can thus be written:

$$p \vee {\sim}p$$

As you will learn later, the propositional form $p \vee {\sim}p$ is called a tautology. In fact, the logical forms of logically true propositions are tautologous.

1.2. Contingent and Logical Falsehood

Some propositions are contingently false (The President of the U.S.A. is a woman), some are logically false (Betty will marry John but she will not marry him). Contingently false statements are false because of how things are in the world. Logically false statements are false necessarily, independently of the facts – they are false in virtue of their logical form. The logical form of the proposition "Betty will marry John but she will not marry him" can also be written as:

$$p \bullet {\sim}p$$

As you will learn later, the propositional form $p \bullet {\sim}p$ is called a contradiction. In fact, the logical forms of logically false propositions are contradictory.

2. Propositional Forms

2.1. Proper Logical Form of a Proposition

Consider the following sentence:

(1) If Betty passes logic then she will get a car, but if she does not then she will not get it.

which can be symbolized thus:

L: Betty passes logic
C: Betty will get a car

[1] $(L \rightarrow C) \bullet (\sim L \rightarrow \sim C)$

The proposition [1] has a certain logical structure, which is captured by the ***propositional form*** (i):

(i) $(p \rightarrow q) \bullet (\sim p \rightarrow \sim q)$

Formula (i) is a propositional form, while formula (1)/[1] is a proposition. Propositional form (i) is the ***proper logical form*** of proposition (1)/[1]. And conversely, proposition (1)/[1] is a ***proper substitution instance*** of propositional form (i).

Definition

A proposition $\mathcal{P}$ is a **proper substitution instance** of a propositional form $\mathcal{F}$ iff $\mathcal{F}$ and $\mathcal{P}$ identical except that that propositional variables in $\mathcal{F}$ have been replaced with propositional constants in $\mathcal{P}$, where (a) same-shaped propositional variables were replaced with same-shaped propositional constants and (b) differently-shaped propositional variables were replaced with differently shaped propositional constants.

Definition

Propositional form $\mathcal{F}$ is a **proper logical form** of proposition $\mathcal{P}$ iff a proposition $\mathcal{P}$ is a proper substitution instance of a propositional form $\mathcal{F}$.

A given propositional form

(i) $(p \rightarrow q) \bullet ({\sim}p \rightarrow {\sim}q)$

is thus the proper logical form of the following propositions:

[1] $(L \rightarrow C) \bullet ({\sim}L \rightarrow {\sim}C)$
[2] $(M \rightarrow N) \bullet ({\sim}M \rightarrow {\sim}N)$
[3] $(A \rightarrow B) \bullet ({\sim}A \rightarrow {\sim}B)$

but it is *not* the proper logical form of the following propositions:

[4] $(L \vee C) \bullet ({\sim}L \vee {\sim}C)$	because (i) and [4] differ not only in that where there are propositional constants in [4] there are propositional variables in (i); the logical constants differ as well!
[5] $(C \rightarrow D) \bullet ({\sim}D \rightarrow {\sim}C)$	because condition (a) is not satisfied
[6] $(A \rightarrow A) \bullet ({\sim}A \rightarrow {\sim}A)$	because condition (b) is not satisfied

2.2. Two Types of Logical Formulas: Propositions vs. Propositional Forms

Every proposition says that something is the case and this is why it can be true or false. Propositions are thus the bearers of truth-values: they are either true or false.

> Propositions state that something is the case. Every proposition is either true or false.

Propositional forms are not the sort of formulas that can be true or false! They are only abstract logical schemata. They don't say anything.

> Propositional forms do not state anything. Propositional forms are neither true nor false.

We will be using the term 'formula' to capture the "mother" category of the two categories of propositions and propositional forms:

formulas

propositions   propositional forms

3. Tautologies, Contradictions, Contingencies

Propositional forms can be: tautologous, contradictory or contingent.

3.1. Tautologies

Some propositional forms are such that no matter what statements you substitute for the propositional variables you will always get a true propositions as a result of the substitution. In other words, some propositional forms have only true substitution instances. Such propositional forms are called tautologies – and the substitution instances of those forms are logically true. Here is an example of such a form:

$$p \vee {\sim}p$$

When you substitute propositions for ‘p’, no matter whether true or false, you would always get a true proposition as a result.

Definition

Tautologies are such propositional forms all of whose substitution instances are true.

3.2. Contradictions

There are other propositional forms with only false substitution instances – no matter what propositions you substitute for the propositional variables, you will always get a false proposition as a result. These propositional forms are called contradictions. Again recall the example of such a proposition:

$$p \bullet {\sim}p$$

Again, it does not matter whether you substitute a true proposition or a false proposition for *p*, the resulting proposition will be false.

Definition

Contradictions are propositional forms all of whose substitution instances are false.

3.3. Contingencies

Finally, there are propositional forms that do not determine the truth-value of their substitution instances – they are called contingencies. When a proposition that has the propositional form of a contingency is true, the proposition is said to be contingently true, when such a proposition is false, it is said to be contingently false. Consider the following propositional form:

$${\sim}p$$

Depending on whether you substitute a false or a true proposition for *p*, the resulting proposition will be true or false, respectively.

Definition

Contingencies are those propositional forms some of whose substitution instances are true and some of whose substitution instances are false

3.4. Properties of Propositional Forms vs. Properties of Propositions

Because propositions and propositional forms are different kinds of entities, they also differ in the properties they have. Propositions can be true or false, moreover they be true either logically or contingently or they can false either logically or contingently. Propositional forms on the other hand, cannot be either true or false. Propositional forms are tautologous, contradictory or contingent.

Moreover, there is a relationship between these properties of propositions and their logical forms, which is captured by the following table:

Propositional form	Propositions that are substitution instances of that form
Tautologous	Logically true
Contingent	Contingently true
	Contingently false
Contradictory	Logically false

4. Truth-Table Definitions of a Tautology, a Contradiction, a Contingency

Logicians have invented the truth-table method for checking whether a propositional form is tautologous, contradictory or contingent. Truth tables for propositional forms allow to determine all the possible truth-values that the substitution instances of those forms can have.

A propositional form that is true in all rows of its truth table is a **tautology**.
A propositional form that is false in all rows of its truth table is a **contradiction**.
A propositional form that is true in at least one row of its truth table and false in at least one row of its truth table is a **contingency**.

We have already said that $p \vee {\sim}p$ is a tautology. In the propositional form $p \vee {\sim}p$ there is only one propositional variable p, which means that it can be substituted either by propositions that are true (considered in row 1 of the truth table) or by propositions that are false (considered in row 2). We thus need to calculate the truth-value of the complex propositions for each of these possibilities (rows):

	p	$p \vee {\sim}p$		
1.	T	T ∨ ~T	T ∨ F	T
2.	F	F ∨ ~F	F ∨ T	T

The propositional form $p \vee {\sim}p$ is a tautology because all of its substitution instances are true, which is shown by the truth table – the propositional form is true in all rows of its truth table.

As a point of terminology, you should remember that the highlighted part of the truth table is called the **truth-table base**. We will say more about how truth-table bases are constructed in §6.

Let's consider an example of a contingency: $p \vee (p \bullet q)$. Now note, however, that there are two propositional variables in that propositional form, viz. p and q. We thus need to consider more possibilities – in fact, four – of truth-value assignments for variables p and q:

	p	q	$p \vee (p \bullet q)$		
1.	T	T	T ∨ (T • T)	T ∨ T	T
2.	T	F	T ∨ (T • F)	T ∨ F	T
3.	F	T	F ∨ (F • T)	F ∨ F	F
4.	F	F	F ∨ (F • F)	F ∨ F	F

The propositional form $p \vee (p \bullet q)$ is a contingency because there are at least one row (in fact there are two such rows – rows 1 and 2) where it is true, and there is at least one row in fact there are two such rows – rows 3 and 4) where it is false.

5. Determining the Properties of Propositions

5.1. Example

Is the following statement contingently or logically true, or contingently or logically false?

(1) Some roses smell nice while some roses smell like dirty socks.

The first thing we do is to symbolize the statement:

N: Some roses smell nice
D: Some roses smell like dirty socks

[1] N • D

The next thing we need to do is to determine what the truth-values of the simple propositions are (here N is true, while D is false), and to determine what the resulting truth-value of the proposition (1) is:

T • F
F

Proposition (1) is false. The question remains whether it is logically or contingently false. To determine that we need to identify the proper logical form of the proposition:

{1} *p* • *q*

and to construct a truth table for the propositional form:

p	*q*	*p* • *q*	
T	T	T • T	T
T	F	T • F	F
F	T	F • T	F
F	F	F • F	F

The propositional form *p* • *q* is a contingency since there is at least one row where the form is true (row 1) and at least one row where it is false (rows 2, 3, 4).

Since proposition (1) is false and its proper logical form is a contingency, we can conclude that proposition (1) is contingently false.

5.2. Example

Is the following statement contingently or logically true, or contingently or logically false?

(3) If logic is not both useful and interesting then logic is either not useful or not interesting.

The first thing we do is to symbolize the statement:

I: Logic is interesting
U: Logic is useful

[3] $\sim(U \bullet I) \rightarrow (\sim U \vee \sim I)$

The next thing we need to do is to determine what the truth-values of the simple propositions are (here I is true, and U is true, too – actually, as you will see, it does not matter what truth-values you'd assign to the simple propositions in this case), and to determine what the resulting truth-value of the proposition (3) is:

$$\sim(T \bullet T) \rightarrow (\sim T \vee \sim T)$$
$$\sim(T) \rightarrow (F \vee F)$$
$$F \rightarrow F$$
$$T$$

Proposition (3) is thus true. The question remains whether it is logically or contingently true. To determine that we need to identify the proper logical form of the proposition:

{3} $\sim(p \bullet q) \rightarrow (\sim p \vee \sim q)$

and to construct a truth table for the propositional form:

p	q	$\sim(p \bullet q) \rightarrow (\sim p \vee \sim q)$			
T	T	$\sim(T \bullet T) \rightarrow (\sim T \vee \sim T)$	$\sim(T) \rightarrow (F \vee F)$	$F \rightarrow (F)$	$\boxed{T}$
T	F	$\sim(T \bullet F) \rightarrow (\sim T \vee \sim F)$	$\sim(F) \rightarrow (F \vee T)$	$T \rightarrow (T)$	$\boxed{T}$
F	T	$\sim(F \bullet T) \rightarrow (\sim F \vee \sim T)$	$\sim(F) \rightarrow (T \vee F)$	$T \rightarrow (T)$	$\boxed{T}$
F	F	$\sim(F \bullet F) \rightarrow (\sim F \vee \sim F)$	$\sim(F) \rightarrow (T \vee T)$	$T \rightarrow (T)$	$\boxed{T}$

The propositional form $\sim(p \bullet q) \rightarrow (\sim p \vee \sim q)$ is a tautology since it is true in all rows of its truth table (this is also why it did not matter much in this case whether you agreed with me that the simple propositions are true).

Since the proper logical form of proposition (3) is a tautology (which means that proposition (3) is true), we can conclude that proposition (3) is logically true.

6. How to Construct a Truth-Table Base

The truth-table base for a propositional form with one variable will be:

p
T
F

and for two variables:

p	*q*
T	T
T	F
F	T
F	F

In general, a truth-table base for a propositional form with *n* variables will have 2^n (2 to the power of *n*, i.e. you need to carry the multiplication $2 \cdot \ldots \cdot 2$, where there are *n* 2s to multiply) rows.

The Algorithm for Constructing a Truth-Table Base

I provide here an algorithm for constructing a truth-table base for an arbitrarily large number of variables. Let us suppose that there are four variables. The first thing to do is to list them in an alphabetical order

p	*q*	*r*	*s*

Then beginning with the rightmost variable (here: *s*) construct a truth table for *s*.

p	*q*	*r*	*s*
			T
			F

Copy the truth-table for *s* beneath itself, and to the left (here: under *r*) type T's next to the original truth table, and F's next to the copied truth table

p	*q*	*r*	*s*
		T	T
		T	F
		F	T
		F	F

When you look at the truth table under *r* and *s*, this is the regular 2-variable truth table. Indeed the next step is to copy the whole thing underneath itself once again, and in the column to the left (here: under *q*) write T's next to the original truth table and F's next to the copied truth table:

p	*q*	*r*	*s*
	T	T	T
	T	T	F
	T	F	T
	T	F	F
	F	T	T
	F	T	F
	F	F	T
	F	F	F

The truth table under *q*, *r* and *s* is a 3-variable truth table. The next step is to copy the whole 3-variable truth table underneath itself once more, and in the column to the left (here: under *p*) write T's next to the original truth table and F's next to the copied truth table:

p	*q*	*r*	*s*
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

The truth table for 4-variables is thus completed. But obviously the same procedure could be continued if there were more than 4 more variables.